STATISTICAL RESEARCH REPORT No 6
Institute of Mathematics September 1978
University of Oslo

ON THE TRANSIENT WAITING TIMES FOR A GI/M/1 PRIORITY QUEUE

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Abstract

In this paper we consider the GI/M/1 queueing model with infinite waiting-room capacity. The customer arriving at $t=0$ will find $k-1$ customers waiting. The latter customers belong to a second priority class, whereas the ones arriving in $[0,\infty)$ belong to a first priority class and have the higher priority. Within each class we have a first-in-first-out queueing discipline. A customer once at the service-point, remains there until his service is completed. Then the next customer for service is the one of highest priority among those queueing.

For this model we derive the transient waiting times for customers belonging to both priority classes. The results are of special interest in appointment systems where customers may not turn up.

GI/M/1 PRIORITY QUEUE ; TRANSIENT WAITING TIMES; NONPREEMPTIVE PRIORITY DISCIPLINE ; APPOINTMENT SYSTEMS

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1. Introduction

Consider the following single server queueing model with infinite waiting-room capacity. Let $(n=0,1,2,\ldots)$

t_n = arrival epoch of the n-th customer arriving in $[0,\infty)$,

where $t_o=0$. Assume the interarrival times to be independent and identically distributed with distribution function $A(t)$ and let the lengths of the service intervals be independent and exponentially distributed with parameter μ. Also the arrival and service processes are independent.

The customer arriving at $t=0$ will find $k-1$ customers waiting. The latter customers belong to a second priority class, whereas the ones arriving in $[0,\infty)$ belong to a first priority class and have the higher priority. Within each class we have a first-in-first-out queueing discipline. The priority discipline is nonpreemptive; i.e. a customer once at the service-point remains there until his service is completed. Then the next customer for service is the one of highest priority among those queueing.

As a motivation for studying the present model consider the following specialization of the arrival pattern above which is realistic when, for example, doctors, dentists or lawyers are consulted. Let the intervals between possible arrivals have fixed length $1/\lambda$ and let the probability of a customer not turning up be $1-p$. Customers turn up or not independently of each other. The number of intervals of length $1/\lambda$ between the arrivals of two customers are then geometrically distributed with parameter p. The $k-1$ customers in the second priority class do not have an appointment, but are allowed to queue up for instance either before the office is opened in the morning or before it is reopened after the lunch break. One is now interested in:

i) waiting times for customers from both priority classes to be not too long

ii) the intitial busy period (starting with k customers in the system) to be not too short.

Small values of k will satisfy i) whereas large values satisfy ii). Defining some optimality criteria Dalen (1976), treating this model, gives optimal values for k, for various values of traffic intensity and p. This work will be presented in a forthcoming joint paper by the present authors. In principal, results are available to give optimal values for k for any specialization of the general model.

In the present paper we will for the general model derive the transient waiting times for customers belonging to both priority classes. The desired results on the busy period follow from Bhat (1967) treating GI/M/1 with batch service. Moreover, our deductions here are mainly based on some expressions given in this paper. However, some of these expressions seem to be lacking some terms. These apparent mistakes we attempt to correct in Appendix I. Furthermore, we have to generalize a result in Takács (1962). This is done in Appendix II.

Finally it should be noted that a predecessor of the present paper, Dalen and Natvig (1978), appeared as a contributed paper at the "Seventh Conference on Stochastic Processes and their Applications". The main steps in the present paper are due to B. Natvig, whereas the forthcoming paper almost entirely is due to G. Dalen.

2. Waiting times for customers from the second priority class

Denote the j-th customer present just after $t=0$ by $\tilde{K}_j$ $(j=1,\dots,k)$, having numbered the customers according to priority. Introduce the random variables (r.v.'s)

$\tilde{U}_j$ = time spent in queue by $\tilde{K}_j$

B_j = length of a busy period starting with j customers in the system.

Since the customer arriving at $t=0$, does not have to wait

$$\tilde{U}_1 = 0.$$

Furthermore, $(j=2,\dots,k)$

$$\tilde{U}_j = B_{j-1}. \tag{2.1}$$

This is true since the service of $\tilde{K}_j$ $(j=2,\dots,k)$, being a member of the second priority class, is not started before the busy period, initiated with the customers $\tilde{K}_1,\dots,\tilde{K}_{j-1}$ in the system, is completed.

Using the notation of Bhat (1967) let

$g_i^{(n)}(t)dt$ = P [Busy period starting with $i+1$ customers in the system terminates in $(t,t+dt)$ after n service completions]

For the case when customers are served one at a time, a corrected version of $g_i^{(n)}$ is given by (AI,5) in Appendix I. Using this expression the probability density function of $\tilde{U}_j$ $(j=2,\dots,k)$, $\tilde{u}_j(t)$, is given by

$$\tilde{u}_j(t) = \sum_{n=j-1}^{\infty} g_{j-2}^{(n)}(t) \qquad t\geq 0. \tag{2.2}$$

We have hence arrived at the distribution for $\tilde{U}_j$ $(j=1,\dots,k)$. The waiting time (including service) for $\tilde{K}_j$ is just the convolution of the distribution of $\tilde{U}_j$ and the exponential distribution regulating the service times.

3. Waiting times for customers from the first priority class

Denote the n-th customer arriving in $[0,\infty)$ by K_n $(n=0,1,\dots)$. Note that $K_o = \tilde{K}_1$, the waiting time of which is treated in the previous section. Introduce the r.v.'s $(n=1,2,\dots)$

U_n = time spent in queue by K_n

Y_n = number of customers from the first priority class present in the system <u>just after</u> the arrival of K_n.

Here and in the following <u>any</u> customer being served is considered as a member of the first priority class. Then

$$P(U_n \le x) = \sum_{r=1}^{n+1} P(U_n \le x | Y_n = r) P(Y_n = r)$$

$$= \sum_{r=1}^{n+1} P(Y_n = r) \int_0^x \frac{(\mu u)^{r-2}}{(r-2)!} \mu e^{-\mu u} du. \tag{3.1}$$

What remains is hence to find

$P(Y_n=r)$; $r=1,\dots,n+1$.

Now following Bhat (1967) define the processes $\{N(t)\}$ and $\{D(t)\}$ by

$N(t)$ = number of arrivals in $(0,t]$

$D(t)$ = number of services in $(0,t]$ if the service process is running without any break.

Let for $n \ge 1$, $A_n(t)$ denote the n-fold convolution of $A(t)$ with itself and let $A_o(t) = 1$, $t \ge 0$; $A_o(t) = 0$, $t<0$. Then obviously $(n=1,2,\dots)$

$$P[N(t)=n] = A_n(t) - A_{n+1}(t) \quad , \quad t>0$$

$$P[D(t)=n] = \frac{(\mu t)^n}{n!} e^{-\mu t} \quad , \quad t>0.$$

Now for $k \geq 1$

$$B_k = \inf\{t \mid k+N(t) - D(t) \leq 0\}$$

$$B_o \overset{\text{def}}{=} 0.$$

We then have the following equivalence ($0 \leq s \leq k-1; n=1,2,\ldots$)

$$B_s \leq t_n < B_{s+1}$$

$$\Updownarrow$$

On the arrival of K_n the service of $\tilde{K}_{s+1}$ has started (and may be finished), whereas $k-1-s$ customers from the second priority class are waiting in the queue.

This implies the following equivalence ($0 \leq s \leq k-1; n=1,2,\ldots$)

$$(k+N(t_n) - D(t_n) = j) \cap (B_s \leq t_n < B_{s+1})$$

$$\Updownarrow \tag{3.2}$$

$$(Y_n = j - (k-1-s)) \cap (B_s \leq t_n < B_{s+1})$$

Since $B_s \leq t_n < B_{s+1}$ and $0 \leq s \leq k-1$, K_n can not find the system empty. Hence in the last statement

$$2 \leq Y_n \leq n+1,$$

which implies that j in (3.2) must satisfy

$$k-s+1 \leq j \leq n+k-s.$$

We now have $(r=1,\ldots,n+1;n=1,2,\ldots)$

$$P(Y_n=r) = \sum_{s=o}^{k-2} P[(Y_n=r) \cap (B_s \leq t_n < B_{s+1})]$$

$$+ P[(Y_n=r) \cap (B_{k-1} \leq t_n)]$$

$$= \sum_{s=o}^{k-2} P[(k+N(t_n)-D(t_n)=r+k-1-s) \cap (B_s \leq t_n < B_{s+1})]$$

$$+ P[(Y_n=r) \cap (B_{k-1} \leq t_n)], \tag{3.3}$$

having applied (3.2). Let us first consentrate on the last addend in (3.3). Introduce the r.v. $(n=1,2,\ldots)$

Q_n = total number of customers present in the system <u>just after</u> the arrival of K_n.

Now

$$B_{k-1} \leq t_n$$

$$\Updownarrow$$

There are no customers from the second priority class waiting in the queue on the arrival of K_n.

Hence $(r=1,\ldots,n+1;n=1,2,\ldots)$

$$P[(Y_n=r) \cap (B_{k-1} \leq t_n)] = P[(Q_n=r) \cap (B_{k-1} \leq t_n)]$$

$$= P(Q_n=r) - P[(Q_n=r) \cap (t_n < B_{k-1})]$$

$$= P(Q_n=r) - \sum_{s=o}^{k-2} P[(Q_n=r) \cap (B_s \leq t_n < B_{s+1})]$$

$$= P(Q_n=r) - \sum_{s=o}^{k-2} P[(k+N(t_n)-D(t_n)=r) \cap (B_s \leq t_n < B_{s+1})]. \tag{3.4}$$

Introduce

$$h(s,j,n) = \begin{cases} P[(k+N(t_n) - D(t_n) = j) \cap (B_s \le t_n < B_{s+1})] \\ \quad \text{for} \quad 0 \le s \le k-2; k-s+1 \le j \le n+k-s; n \ge 1 \\ 0 \text{ otherwise.} \end{cases} \tag{3.5}$$

Applying this notation and inserting (3.4) into (3.3), we get $(r=1,\dots,n+1;n=1,2,\dots)$

$$P(Y_n=r) = \sum_{s=o}^{k-2} [h(s,r+k-s-1,n) - h(s,r,n)] + P(Q_n=r). \tag{3.6}$$

Now the distribution of Q_n for the case $k=1$ (one customer in the system at $t=0$) is found by elegant combinatorial methods in Takács (1962). These methods are generalized in Appendix II to give $P(Q_n=r)$ for an arbitrary $k \ge 1$.

Hence to establish $P(Y_n=r)$ given by (3.6) it remains to arrive at $h(s,j,n)$ given by (3.5) for $0 \le s \le k-2; k-s+1 \le j \le n+k-s; n \ge 1$. We now have for these values of s, j and n

$$\begin{aligned} h(s,j,n) &= \int_{t=o}^{\infty} d_t P[(k+N(t_n)-D(t_n)=j) \cap (B_s \le t_n < B_{s+1}) \cap (t_n \le t)] \\ &= \int_{t=o}^{\infty} d_t P[(k+N(t_n)-D(t_n)=j) \cap (t_n < B_{s+1}) \cap (t_n \le t)] \\ &\quad - \int_{t=o}^{\infty} d_t P[(k+N(t_n)-D(t_n)=j) \cap (t_n < B_s) \cap (t_n \le t)] \\ &= \int_{t=o}^{\infty} d_t P[(s+1+N(t_n)-D(t_n)=s+1+j-k) \cap (t_n < B_{s+1}) \cap (t_n \le t)] \\ &\quad - \int_{t=o}^{\infty} d_t P[(s+N(t_n)-D(t_n)=s+j-k) \cap (t_n < B_s) \cap (t_n \le t)]. \end{aligned}$$

Introduce

$$h_1(s,j,n) = \begin{cases} \int_{t=o}^{\infty} d_t P[(s+1+N(t_n)-D(t_n)=j) \cap (t_n<B_{s+1}) \cap (t_n\leq t)] \\ \quad \text{for} \quad 0\leq s\leq k-2; 1\leq j\leq n+1 \;;\; n\geq 1 \\ 0 \quad s = -1 \end{cases} \tag{3.7}$$

Hence for $0\leq s\leq k-2$; $k-s+1\leq j\leq n+k-s$; $n\geq 1$

$$h(s,j,n) = h_1(s,s+1+j-k,n) - h_1(s-1,s+j-k,n) \tag{3.8}$$

Now $h_1(s,j,n)$ for $0\leq s\leq k-2$; $2\leq j\leq n$; $n\geq 2$ is found by applying the relations (9), (20), (24) in Bhat (1967). (Note that the j in his (20) corresponds to our $j-1$. For a short introduction to the notation in this paper see Appendix I.)

We get $(0\leq s\leq k-2 \;; 2\leq j\leq n \;;\; n\geq 2)$

$$\begin{aligned} h_1(s,j,n) = &\int_0^{\infty} \frac{(\mu t)^{n+s-j+1}}{(n+s-j+1)!} e^{-\mu t}\, d\, A_n(t) - \\ &\int_0^{\infty} \sum_{m=1}^{n-j+1} \int_0^t \frac{(\mu(t-\tau))^{n-m-j+1} (\mu\tau)^{m+s} (j-1)}{(n-m-j+1)!\; (m+s)!\; (n-m)} e^{-\mu t} dA_m(\tau) dA_{n-m}(t-\tau) \end{aligned} \tag{3.9}$$

Now for $0\leq s\leq k-2$; $n\geq 1$

$$h_1(s,1,n) = 0, \tag{3.10}$$

since the events $(s+1+N(t_n)-D(t_n) = 1)$ and $(t_n<B_{s+1})$ are disjoint. Furthermore, for these values of s and n

$$\begin{aligned} &h_1(s,n+1,n) = \\ &\int_0^{\infty} d_t P[(s+1+N(t_n) + D(t_n) = n+1) \cap (t_n<B_{s+1}) \cap (t_n\leq t)] \\ &= \int_0^{\infty} d_t P[(s+1+N(t_n) + D(t_n) = n+1) \cap (t_n\leq t)] \\ &= \int_0^{\infty} \frac{(\mu t)^s}{s!} e^{-\mu t}\, d\, A_n(t). \end{aligned} \tag{3.11}$$

Hence $h_1(s,j,n)$ for $0 \leq s \leq k-2$; $1 \leq j \leq n+1$; $n \geq 1$ is given by (3.9)-(3.11). The distribution of U_n can then be set up by combining (3.1), (3.5)-(3.11) and the distribution of Q_n for $k \geq 1$ from Appendix II. Again the waiting time (including service) for K_n is straightforward.

APPENDIX I - An attempt to correct some apparent mistakes in Bhat (1967)

The contents of this Appendix is taken from Natvig (1975). Since the paper Bhat (1967) is essential for the arguments of the present paper, and the results given there for the busy period for GI/M/1 are crucial when attempting to obtain optimal values for k, we will reproduce the whole of these corrections. In the case where the notation of Bhat (1967) differs from ours, we will make this clear and use his notation in the present Appendix.

The model considered in Bhat (1967) is GI/M/1 with batch service. The service intensity is λ and the probability distribution of the size of the service batch is b_r, $r \geq 1$. Arriving customers join the batch in service till it is full, without affecting the service time. Let $b_r^{(k)}$ be the k-fold convolution of b_r with itself and $b_r^{(o)} = 0$, $r \neq 0$, $b_o^{(o)} = 1$. The busy period, T_i, is initiated by i waiting customers just before an arrival at $t=0$; i.e. $(i+1)$ corresponds to k and T_i to B_k in our notation.

Having defined $G_{ij}^{(n)}(t)$ earlier in the paper he should in connection with Lemma 2 set up

$$G_{ij}^{(n)}(t) = P(i+1+N(t)-D(t)=j,\ T_i>t,\ N(t)=n), \tag{AI.1}$$

the 1 in $(i+1)$ missing without, however, affecting the results.

Introducing $g_i^{(n)}(t)$, as mentioned in Section 2 of the present paper, we have

$$g_i^{(n)}(t) = \sum_{r=1}^{n} G_{ir}^{(n-i-1)}(t)\lambda \sum_{k=r}^{\infty} b_k \ , \ n>i \tag{AI.2}$$

the author wrongly stopping the summation at $r = n-1$ both in his (14) and (28), respectively treating $i=0$ and general i. This implies that the explicit expression for $g_o^{(n)}(t)$ given in (15) is lacking the following term corresponding to no departures in $(0,t)$, the service of the whole batch of n being completed in $[t,t+dt]$

$$e^{-\lambda t}\left\{\int_0^t dA_{n-1}(\tau)[1-A(t-\tau)]\right\}\lambda(1-\sum_{k=1}^{n-1} b_k).$$

For general i no corresponding expression is given. When customers are served one at a time this mistake has had no influence on the results arrived at.

For $i>1$ (AI.1) is established indirectly by calculating

$$P(i+1+N(t)-D(t) = j,\ T_i<t,\ N(t) = n). \tag{AI.3}$$

We now have

$$T_i = \inf\{t|i+1+N(t) - D(t) \leq 0\},$$

the 1 again missing. The process $i+1+N(t) - D(t)$ is non-Markovian; however, the points at which it is zero and we are just to have an arrival are points of regeneration (the author is somewhat unprecise here). To establish (AI.3) the last of these points (τ) is considered. When $j=1$, the arrival may be the n-th and last one giving the following forgotten contribution to $G_{i1}^{(n)}(t)$

$$- \sum_{k=1}^{n+i} \int_0^t e^{-\lambda\tau}(\lambda\tau)^k/k!\; b_{n+1}^{(k)} dA_n(\tau) e^{-\lambda(t-\tau)}(1-A(t-\tau)). \qquad \text{(AI.4)}$$

According to (AI.2) this will affect $g_i^{(n)}(t)$ both when customers are served in batches and one at a time. In the latter case the correct version of $g_i^{(n)}(t)$ is

$$g_i^{(n)}(t) = e^{-\lambda t} \frac{\lambda^n}{(n-1)!} t^{n-1}[A_{n-i-1}(t)-A_{n-i}(t)]$$

$$- \sum_{m=1}^{n-i-2} \frac{\lambda^n e^{-\lambda t}}{m!(n-m-1)!} \int_{u=0}^{t} \int_{\tau=0}^{u} \tau^{n-m-1}(t-\tau)^{m-1}(t-u)[1-A(t-u)]dA_m(u-\tau)dA_{n-i-1-m}(\tau) \qquad \text{(AI.5)}$$

$$- e^{-\lambda t}\frac{\lambda^n}{(n-1)!} \int_0^t \tau^{n-1} dA_{n-i-1}(\tau)(1-A(t-\tau)),$$

having replaced the incorrect $dA_{n-i-1}(u-\tau)$ by $dA_m(u-\tau)$ and subtracted the contribution corresponding to (AI.4).

The author finally deduces the joint distribution, $R_i^{(n)}(t)$, of the length of and the number of arrivals in a busy cycle (including the one at $t=0$). In his explicit expression for $dR_o^{(n+1)}(t)$, however, the summation in the last term shall start from $r=0$ instead of $r=1$.

APPENDIX II - A generalization of a result in Takács (1962)

In this appendix we will derive the distribution of Q_n, defined in Section 3, for an arbitrary $k \geq 1$, thus generalizing the result in Takács (1962) treating the case $k=1$. Following the notation in this paper introduce the r.v.

$$\xi_n = Q_n - 1,$$

i.e. the total number of customers present in the system just before the arrival of K_n. Denote by ν_n $(n=1,2,\ldots,)$ the number of events in the $\{D(t)\}$ process in the interval $[t_{n-1}, t_n)$. Then $\{\nu_n\}$ is a sequence of identically distributed, mutually independent r.v.'s with distribution

$$q(j) = P(\nu_n = j) = \int_0^\infty e^{-\mu x} \frac{(\mu x)^j}{j!} \, d\, A(x)$$

and n-fold convolution, $n \geq 1$

$$q_n(j) = P(\nu_1 + \ldots + \nu_n = j) = \int_0^\infty e^{-\mu x} \frac{(\mu x)^j}{j!} \, d\, A_n(x).$$

By a trivial extension of the argument leading to (8) in Takács (1962) ξ_n has the same distribution as

$$\tilde{\xi}_n = \max\{k-1+n-\nu_1-\ldots-\nu_n, n-1-\nu_1\ldots-\nu_{n-1}, \ldots, 1-\nu_1, 0\}.$$

Denote $\tilde{\xi}_n$ by $\tilde{\xi}^*_n$ when $k=1$. Hence

$$\tilde{\xi}_n = \max\{k-1+n-\nu_1-\ldots-\nu_n, \tilde{\xi}^*_{n-1}\}. \tag{AII.1}$$

The distribution of $\tilde{\xi}^*_n$ $n \geq 1$ is given by Theorem 1 in Takács (1962) whereas $\tilde{\xi}^*_o = 0$. Now introduce

$$p_m^{(n)} = P(\xi_n \geq m) = P(\tilde{\xi}_n \geq m). \tag{AII.2}$$

Obviously

$$\begin{aligned} p_m^{(n)} &= 0 \qquad m \geq n+k \; ; \; n \geq 1 \\ p_o^{(n)} &= 1 \qquad n \geq 1. \end{aligned} \tag{AII.3}$$

Assume $n \leq m \leq n+k-1$. Since $\tilde{\xi}^*_{n-1} \leq n-1$, we have from (AII.1)

$$\begin{aligned} p_m^{(n)} &= P(k-1+n-\nu_1-\ldots-\nu_n \geq m) \\ &= \sum_{l=o}^{k-1+n-m} q_n(l) \qquad n \leq m \leq n+k-1 \; ; \; n \geq 1. \end{aligned} \tag{AII.4}$$

What remains is the case $1 \leq m \leq n-1$.

From (AII.1)

$$
\begin{aligned}
P_m^{(n)} &= P(\tilde{\xi}^*_{n-1} \geq m) + P(k-1+n-\nu_1-\ldots-\nu_n \geq m) \\
&\quad - P[(\tilde{\xi}^*_{n-1} \geq m) \cap (k-1+n-\nu_1-\ldots-\nu_n \geq m)] \\
&= \sum_{j=m}^{n-1} \frac{m}{j} q_j(j-m) + \sum_{l=o}^{k-1+n-m} q_n(l) \\
&\quad - P[(\tilde{\xi}^*_{n-1} \geq m) \cap (k-1+n-\nu_1-\ldots-\nu_n \geq m)],
\end{aligned}
\tag{AII.5}
$$

having applied the mentioned theorem. Consider the last term. From (9), (10), (11) in Takács (1962) we have the following equivalence

$$
\tilde{\xi}^*_{n-1} \geq m
$$

$$
\Updownarrow
$$

$$
\bigcup_{j=m}^{n-1} \left\{(j-\nu_1-\ldots-\nu_j=m) \cap (\nu_{r+1}+\ldots+\nu_j<j-r, 1 \leq r<j)\right\}.
$$

Since the events in the union are shown to be disjoint, we get

$$
\begin{aligned}
&P[\tilde{\xi}^*_{n-1} \geq m \cap (k-1+n-\nu_1-\ldots-\nu_n \geq m)] \\
&= \sum_{l=o}^{k-1+n-m} \sum_{j=m}^{n-1} P(\nu_{j+1}+\ldots+\nu_n = l-j+m) \\
&\circ\ P(\nu_{r+1}+\ldots+\nu_j<j-r, 1 \leq r<j \mid \nu_1+\ldots+\nu_j=j-m) \\
&\circ\ P(\nu_1+\ldots+\nu_j=j-m) \\
&= \sum_{l=j-m}^{k-1+n-m} \sum_{j=m}^{n-1} q_{n-j}(l-j+m) q_j(j-m) \frac{m}{j},
\end{aligned}
$$

having applied Lemma 1 in Takács (1962) exactly as is done there for the case $k=1$. Hence from (AII.5)

$$P_m^{(n)} = \sum_{l=o}^{k-1+n-m} q_n(l) + \sum_{j=m}^{n-1} \frac{m}{j} q_j(j-m) \left[1 - \sum_{l=o}^{k-1+n-j} q_{n-j}(l)\right], \qquad \text{(AII.6)}$$

$$1 \leq m \leq n-1 \ ; \ n \geq 2.$$

The distribution of Q_n is now given by $(1 \leq r \leq n+k \ ; \ n \geq 1)$

$$P(Q_n = r) = P(\xi_n = r-1) = P_{r-1}^{(n)} - P_r^{(n)},$$

where $P_m^{(n)}$; $m \geq 0$, $n \geq 1$ is given by (AII.3), (AII.4), (AII.6).

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